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Minimal surfaces with self-intersections along straight lines. I. Derivation and properties

E. Koch* and W. Fischer

Institut für Mineralogie, Petrologie und Kristallographie der Universität Marburg, Hans-Meerwein-Strasse, D-35032 Marburg, Germany. E-mail: kochelke@mailer.uni-marburg.de

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Abstract

A special kind of three-periodic minimal surface has been studied, namely surfaces that are generated from disc-like-spanned skew polygons and that intersect themselves exclusively along straight lines. A new procedure for their derivation is introduced in this paper. Several properties of each such surface may be deduced from its generating polygon: the full symmetry group of the surface, its orientability, the symmetry group of the oriented surface, the pattern of self-intersections, the branch points of the surface, the symmetry and periodicity of the spatial subunits demarcated by the surface, and the Euler characteristics both of the surface and of the spatial subunits. The corresponding procedures are described and illustrated by examples.

1. Introduction

In the past, mainly those three-periodic minimal surfaces that are free of self-intersections, *i.e.* that are embedded in R^3, have been studied. A considerable number of such surfaces have been derived and described in some detail (*cf. e.g.* Cvijović & Klinowski, 1992*a*,*b*,*c*, 1993; Fischer & Koch, 1987, 1989*a*,*b*,*c*, 1990, 1992; Fogden, 1993, 1994; Fogden & Hyde, 1992*a*,*b*; Hyde, 1989; Hyde & Andersson, 1984; Karcher, 1989; Karcher & Polthier, 1990, 1996; Koch & Fischer, 1988, 1989*a*,*b*, 1990, 1993*a*,*b*; Lidin & Hyde, 1987; Mackay, 1985; Neovius, 1883; Schoen, 1970; Schwarz, 1890; Stessmann, 1934). In contrast, three-periodic minimal surfaces with self-intersections have attracted little attention, probably because there exists an infinite variety of such surfaces, showing in most cases too many self-intersections to be of crystallographic interest. Among these, however, a special set stands out, namely those three-periodic minimal surfaces that intersect themselves exclusively along straight lines. Such surfaces are relatively rare and may be derived using crystallographic methods. Their surprising properties will be discussed in this paper and those known so far will be described in succeeding publications.

Each three-periodic minimal surface shows the symmetry of some space group G and, as a consequence, all symmetry operations $g_i \in G$ map the surface onto itself. Among these symmetries, twofold rotations (and mirror reflections) play a special role with respect to minimal surfaces, as has been proved by Schwarz (1894): every straight line running within a minimal surface is a twofold rotation axis of this surface. Three-periodic minimal surfaces containing straight lines, therefore, have been given a special name: spanning minimal surfaces (Fischer & Koch, 1996*b*).

Each three-periodic minimal surface with symmetry G can be subdivided into surface patches, *i.e.* finite congruent parts with the property that the entire infinite surface may be generated by continuing a given surface patch by applying the symmetry operations of G. This continuation process is especially simple for spanning minimal surfaces because a surface patch with a boundary that is formed (partly or as a whole) by straight lines may be used.

A special kind of spanning minimal surface may be derived as follows: a skew polygon, all edges of which are formed by evenfold rotation axes of some space group, is spanned disc-like by a patch of a minimal surface. Its existence is guaranteed without individual proof owing to the general solution of the plateau problem for spanning frames by minimal surfaces [*cf.* textbooks on minimal surfaces, such as those by Nitsche (1989) and Dierkes *et al.* (1992)]. The original surface patch may be continued to an infinite three-periodic surface with the aid of the twofold rotations around the polygon edges and the products of these rotations. The original polygon is called the generating polygon of the spanning minimal surface.

A minimal surface derived in this way may either be free of self-intersections or intersect itself in a more or less complicated pattern. All intersection-free minimal surfaces that may be generated with the aid of skew polygons have been derived previously (Fischer & Koch, 1987; Koch & Fischer, 1988). In the following, self-intersecting surfaces of such a kind will be considered, but only those that intersect themselves exclusively along straight lines (referred to as 'minimal surfaces with straight self-intersections').

2. Derivation

Any skew polygon generating a three-periodic minimal surface with or without self-intersection can be formed by the evenfold rotation axes of one of the following

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space groups: $P222$, $C222$, $F222$, $I222$, $P422$, $P42_12$, $P4_222$, $P4_22_12$, $I422$, $I4_122$, $P622$, $P6_{2,4}22$, $P432$, $P4_232$, $F432$, $F4_132$, $I432$, $P4_{1,3}32$, $I4_132$.

A number of such polygons have been examined. First, each of them has been compared with the known generating polygons of intersection-free minimal surfaces. Then, for each remaining polygon, it had to be checked, individually, whether or not the self-intersections of the spanning minimal surface form not only straight but also curved lines. In some cases, this check was complicated. It was made on the basis of geometrical inspection of the surface patch under consideration and of its neighbouring ones. In this way, about 25 minimal surfaces with straight self-intersections have been derived [most of them have been described earlier (*cf.* Fischer & Koch, 1996*a*,*b*)]. No general rule could be found which property of the polygon guarantees that the generated infinite surface does not intersect itself along curved lines as well as straight lines.

For this reason, a new procedure has been applied: the search has been confined to skew polygons (without nodes), the edges of which run totally on the surface of a convex asymmetric unit of one of the space groups listed above. Each disc-like surface patch (*i.e.* each surface patch with the topological properties of a disc) spanning such a skew polygon lies completely inside this asymmetric unit and, therefore, self-intersection of the surface can only take place along the polygon edges, *i.e.* no complicated individual proof is necessary in these cases. For the space groups listed above, the different convex asymmetric units have been inspected and, for each of them, all skew polygons have been studied that are formed by evenfold rotation axes on the surface of the asymmetric unit. In this way, about 50 additional three-periodic minimal surfaces with straight self-intersections could be derived. It has to be stated, however, that some of the surfaces known before cannot be found in this way. Accordingly, the current list is most probably not complete, in contrast to the corresponding situation in the absence of self-intersections (Fischer & Koch, 1987; Koch & Fischer, 1988).

Due to the fact that spanning of a skew polygon by a part of a minimal surface is guaranteed, irrespective of the shape of the polygon, all these surfaces may be deformed by permitted variation of the metrical parameters. Each 'surface', therefore, represents, strictly speaking, a family of surfaces.

3. Properties

Visualizing a three-periodic minimal surface is often more difficult and the construction of helpful models is more laborious if it intersects itself than if it does not. It turned out, however, that many of the properties of an infinite surface may be derived from its generating polygon by abstract procedures.

3.1. *Full symmetry group G of a minimal surface*

Let P be a skew polygon formed by evenfold rotation axes of some space group G_1. This polygon may be disc-like spanned by a patch of a minimal surface with symmetry G. In most but not all cases, G and G_1 will be identical. G may be calculated as follows. The twofold rotations around the edges of P generate a space group G_2 which either may be identical to G_1 or may be a true subgroup of G_1, *i.e.* $G_2 \subset G_1$. In the latter case, the polygon P is also formed by rotation axes of G_2. If the point-group symmetry of P, regarded by itself, is not higher than 1, G_2 describes the full symmetry of the three-periodic surface, *i.e.* $G_2 = G$. Otherwise, the symmetry operations of the point group of P are additional generators of G, *i.e.* G is a true supergroup of G_2: $G_2 \subset G$.

Example 1. $G_1 = I432$; vertices of the generating polygon: 000, $\frac{1}{2}00$, $\frac{1}{2}\frac{1}{4}0$, $\frac{1}{4}\frac{1}{4}\frac{1}{4}$, $\frac{1}{4}\frac{1}{2}0$, $0\frac{1}{2}0$; polygon edges: $2(x, 0, 0)$, $2(\frac{1}{2}, y, 0)$, $2(x, \frac{1}{4}, \frac{1}{2} - x)$, $2(\frac{1}{4}, y, \frac{1}{2} - y)$, $2(x, \frac{1}{2}, 0)$, $2(0, y, 0)$. The corresponding rotations generate the group $G_2 = P4_232$, *i.e.* $G_2 \subset G_1$. The skew polygon is symmetrical with respect to a diagonal mirror plane $..m$, *i.e.* $m(x, x, z)$ is an additional generator of the full symmetry group $G = Pn\bar{3}m \supset P4_232$ of the minimal surface.

3.2. *Orientability*

Infinite intersection-free minimal surfaces are necessarily orientable, whereas minimal surfaces with self-intersections very often turned out to be non-orientable. Each orientable surface has two sides, which may be coloured differently. On the contrary, a non-orientable surface has, like a Möbius strip, only one side. Each three-periodic minimal surface with disc-like surface patches contains closed rings of these patches. If the surface is non-orientable, then there must exist such rings which are twisted like a Möbius strip.

In the case of a spanning minimal surface generated by a skew polygon (*cf.* §2), each surface patch is related to its neighbouring ones by twofold rotations. Therefore, a closed ring consisting of an odd number of surface patches is necessarily twisted and non-orientable, and consequently must be part of a non-orientable surface. A closed ring with an even number of surface patches, however, may belong either to an orientable or to a non-orientable surface. As a result, the occurrence of an odd-membered ring of surface patches is a necessary and sufficient condition for the non-orientability of a three-periodic surface.

The search for odd-membered rings may be performed by looking for odd products of those twofold rotations that map neighbouring surface patches onto one another. The infinite surface is non-orientable if there exists such an odd-numbered product equal to identity. The number l of surface patches in the shortest odd-membered ring may be used to characterize the non-orientable spanning surface. In practice, the search for odd-membered rings has been performed with the aid of a small PC program which calculates the products of twofold rotations.

3.3. *Symmetry group S of an oriented surface*

The symmetry of any orientable spanning minimal surface may be described by a pair G–S of space groups, regardless of whether or not the surface shows self-intersections. The complete symmetry group G of the surface consists of all symmetry operations mapping the surface onto itself. S is a subgroup of G with index 2 and describes the symmetry of the oriented surface, *i.e.* the symmetry of the surface with both sides coloured differently.

Each spanning minimal surface has twofold rotation axes running within the surface. The respective twofold rotations map the surface onto itself but interchange its two sides and, therefore, belong to G but not to S.

In the case of a minimal surface generated by a skew polygon, S is uniquely determined as that subgroup of G with index 2 that contains none of the twofold rotation axes forming the polygon edges. If, however, no such subgroup of G exists for a given minimal surface, this is another criterion for the non-orientability of that surface.

Example 2. $G = P6/mmm$; vertices of the generating polygon: 000, 100, $10\frac{1}{2}$, $\frac{2}{3}\frac{1}{3}\frac{1}{2}$, $00\frac{1}{2}$; polygon edges: $2(x, 0, 0)$, $2(1, 0, z)$, $2(1 - x, x, \frac{1}{2})$, $2(2x, x, \frac{1}{2})$, $2(0, 0, z)$. Among all the subgroups of $P6/mmm$ with index 2 there exists only one, namely $P6_3/mmc$ ($c' = 2c$), that contains none of these twofold axes. The three-periodic minimal surface generated by the given polygon, therefore, is an orientable one.

Example 3. $G = P432$; vertices of the generating polygon: 000, $\frac{1}{2}\frac{1}{2}0$, $\frac{1}{2}00$, $\frac{1}{2}\frac{1}{2}\frac{1}{2}$, $0\frac{1}{2}0$; polygon edges: $2(x, x, 0)$, $2(\frac{1}{2}, y, 0)$, $2(\frac{1}{2}, y, y)$, $2(x, \frac{1}{2}, x)$, $2(0, y, 0)$. $P432$ has three subgroups with index 2: $P23$, $F432$ ($a' = 2a$) with origin at Wyckoff position a of $P432$, and $F432$ ($a' = 2a$) with origin at Wyckoff position b of $P432$. The axes $2(\frac{1}{2}, y, 0)$ and $2(0, y, 0)$ are preserved in $P23$. The first subgroup $F432$ keeps the axes $2(x, x, 0)$ and $2(0, y, 0)$, the second one $2(\frac{1}{2}, y, 0)$ and $2(\frac{1}{2}, y, y)$. As a consequence, the generated three-periodic minimal surface is non-orientable.

3.4. *Polygon edges with self-intersections*

A minimal surface with straight self-intersections normally does not intersect itself along all edges of its generating polygon. In order to decide whether or not self-intersection occurs along a certain polygon edge, one has to calculate the number n of surface patches sharing that edge. If n equals 2, there is no self-intersection. If n equals 4, 6, 8 or 12, then two, three, four or six pieces of the surface, respectively, intersect each other in the edge under consideration.

To calculate n, the length of the edge under consideration (eventually together with other ones from the same Wyckoff position) has to be divided by the length of all rotation axes of the corresponding Wyckoff position referred to one unit cell of G. The product of this fraction and the number of surface patches per unit cell results in n.

Example 4. $G = P422$; vertices of the generating polygon: 000, 100, $10\frac{1}{2}$, $\frac{1}{2}\frac{1}{2}\frac{1}{2}$, $\frac{1}{2}0\frac{1}{2}$, $00\frac{1}{2}$. There exist eight surface patches per unit cell of $P422$. Polygon edge 1: $2(x, 0, 0)$; Wyckoff position $4l$ .2.; $n_1 = \frac{1}{2} \times 8 = 4$; two pieces of the surface intersect in edge 1. Polygon edges 2 and 6: $2(1, 0, z)$, $2(0, 0, z)$; $2g$ 4..; $n_2 = n_6 = 1 \times 8 = 8$; four pieces of the surface intersect in edges 2 and 6. Polygon edge 3: $2(1 - x, x, \frac{1}{2})$; $4k$ ..2; $n_3 = \frac{1}{4} \times 8 = 2$; no self-intersection. Polygon edge 4: $2(\frac{1}{2}, y, \frac{1}{2})$; $4m$ .2.; $n_4 = \frac{1}{4} \times 8 = 2$; no self-intersection. Polygon edge 5: $2(x, 0, \frac{1}{2})$; $4n$ .2.; $n_5 = \frac{1}{4} \times 8 = 2$; no self-intersection.

If an edge of a generating polygon is formed by a fourfold or by a sixfold rotation axis of G, the generated surface must intersect itself along this edge. In the case of a fourfold rotation axis, two, four or even more pieces of the infinite surface intersect and the surface is necessarily non-orientable. In the case of a sixfold rotation axis, at least three pieces intersect. The corresponding surface may be orientable or non-orientable, examples having been found for both situations.

3.5. *Flat points and branch points*

For any point on a minimal surface, the defining condition $H = \frac{1}{2}(k_1 + k_2) = 0$ is fulfilled, where H is the mean curvature and k_1 and k_2 are the two main curvatures of the surface at that point. Normally, this means $k_1 = -k_2 \neq 0$. If, however, $k_1 = k_2 = 0$ at some point, such a special point is called a 'flat point' of the surface. It has the characteristic property as follows (*cf.* Hyde, 1989; Koch & Fischer, 1990).

Let $\mathbf{n}_0$ be the vector normal to the surface at a flat point P_0, and let P be a second point near P_0 with normal vector $\mathbf{n}$. If P is moved on the surface once around P_0, $\mathbf{n}$ rotates r times ($r > 1$) around $\mathbf{n}_0$. The order of the flat point $\beta = r - 1$ is a measure of the 'degree of flatness' of the surface at the point P_0. β may take any positive integer value, but so far only values up to 4 have been observed for three-periodic minimal surfaces.

A three-periodic minimal surface necessarily has flat points, independently of the existence of self-intersections. In contrast, a 'branch point' can occur only on a self-intersecting surface. It is characterized as follows.

Let P_0 be a branch point with normal vector $\mathbf{n}_0$ and P a point in the neighbourhood of P_0 with normal vector $\mathbf{n}$.

Then, in order to rotate $\mathbf{n}$ once around $\mathbf{n}_0$, P must be moved around P_0 more than once, say u times (*cf.* Fig. 1). The order of the branch point $\gamma = u - 1$ may take any positive integer value in principle, but so far only branch points of order 1 have been observed on three-periodic minimal surfaces.

For spanning minimal surfaces generated by skew polygons, $\gamma = 1$ is the only possible value. Such a branch point results whenever a generating polygon has a vertex angle of 120° (*cf.* Fig. 1). Then the surface intersects itself along each of the twofold axes through the 120° vertex, but only on one side of the vertex, whereas on its other side these axes are not contained in the surface. Such a vertex has at least site symmetry 32. Branch points of order 1 have been found on orientable as well as on non-orientable three-periodic minimal surfaces.

3.6. *Spatial subunits*

Any three-periodic minimal surface subdivides R^3 into spatial subunits. In the case of a surface without self-intersections, there always exist two infinite three-periodic subunits which interpenetrate each other, *i.e.* the two labyrinths of the surface. These labyrinths are necessarily congruent for a spanning minimal surface.

On the contrary, self-intersecting three-periodic minimal surfaces give rise to a variety of spatial subunits, which may differ in their periodicity and their connectivity. The symmetry group U of such a subunit is always a subgroup of the symmetry group G of the surface. Generally, all spatial subunits are symmetrically equivalent with respect to G and their subgroups U_i are either identical or they are conjugate subgroups of G. In the special case, however, that all edges of the generating polygon give rise to self-intersections of the surface, two kinds of spatial subunits exist which are not congruent and of which the symmetry groups are not conjugate in G (*cf.* Fischer & Koch, 1996*a*). For minimal surfaces derived with the aid of skew polygons, the following kinds of subunits have been observed so far.

In most cases a self-intersecting surface subdivides R^3 into two labyrinths which interpenetrate each other like the labyrinths of an intersection-free surface [*cf.* the non-orientable minimal surface described by Schoen (1970)]; but in contrast to these, the labyrinths of a self-intersecting surface may have 'dead ends'. Furthermore, a self-intersecting surface may also subdivide R^3 into four or eight congruent labyrinths. Whether or not other numbers of labyrinths may also occur is not known at present. The symmetry group U of a labyrinth is a space group, and the index of U in G gives the (finite) number of labyrinths.

Some self-intersecting minimal surfaces subdivide R^3 into (an infinite number of) two-periodic spatial subunits, so-called 'flat labyrinths'. Mostly, all flat labyrinths of a surface run parallel to one another, but there exist also a few surfaces of which the flat labyrinths are distributed among two sets. Then, all labyrinths of one set are running parallel to one another, whereas two labyrinths of different sets are perpendicular. In any case, all flat labyrinths are congruent and the symmetry group of a flat labyrinth is a layer group.

One-periodic spatial subunits, so-called 'tubes', have also been observed. Their symmetry groups are crystallographic rod groups. All the tubes of a surface may either run parallel to one another or they may be distributed among sheets of parallel tubes with tubes of neighbouring sheets pointing in different directions. There exists also a minimal surface with two symmetrically inequivalent kinds of tubes.

Spatial subunits without periodicity are finite ones that may be regarded as 'polyhedra' with curved faces but straight edges. In all but one such case, the corresponding minimal surface subdivides R^3 into two kinds of polyhedral subunits with different sizes and different point-group symmetries U.

In addition, three minimal surfaces have been found that subdivide R^3 simultaneously into one-periodic infinite tubes and into finite polyhedra, *e.g.* the *WI*-10 surface (Fischer & Koch, 1996*a*).

As visualizing a spatial subunit without use of a model may also be difficult, a procedure has been derived to generate the symmetry group of a subunit. For this, each edge of the generating polygon has to be assigned to one of three types (I, II and III) defined as follows. A polygon edge without self-intersection belongs to type I. A polygon edge with self-intersection is shared by more than two surface patches, but only two of them are adjacent to the subunit under consideration. They are not mapped onto each other by the twofold rotation corresponding to that edge, but by another symmetry operation of G. Accordingly, symmetrically equivalent

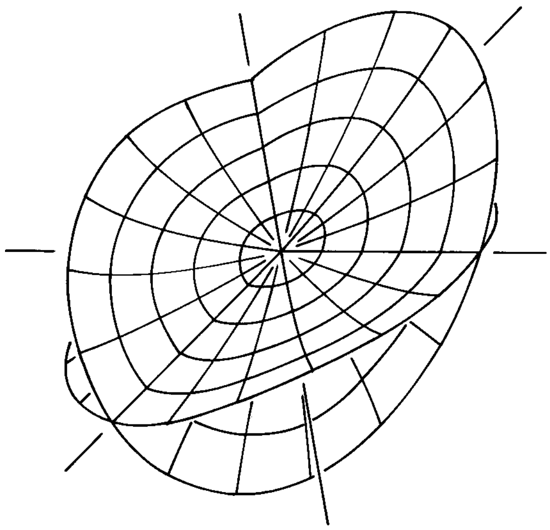

Fig. 1. Neighbourhood of a branch point with site symmetry 32.

sides of the two patches may either be turned towards the regarded spatial subunit (edge of type II) or not (edge of type III).

If a surface patch has an edge of type II, this edge plays a different role with respect to the spatial subunit on one side of that patch or on its other side. In order to take care of both symmetry situations, the patch has to be replaced by a double patch, *i.e.* the union of two neighbouring patches sharing an edge of type I.

A set of generators of the symmetry group U of a spatial subunit is formed by: (a) the symmetry operations of the point group of the original surface patch; (b) all products $g_i g_j$ of two twofold rotations g_i and g_j around edges of type I of the regarded (double) patch; (c) all symmetry operations mapping the regarded double patch onto a neighbouring one if the shared edge is of type II; (d) all ordered products $h_i g_j$ of two symmetry operations, where g_j is a twofold rotation around an edge of type I and h_i maps the regarded (double) patch onto a neighbouring one with a common edge of type III.

Example 5. $G = P422$; vertices of the generating polygon: 000, $\frac{1}{2}00$, $\frac{1}{2}0\frac{1}{2}$, $\frac{1}{2}\frac{1}{2}\frac{1}{2}$, $\frac{1}{2}\frac{1}{2}0$. Four edges belong to type I; the corresponding twofold rotations are: $g_1 = 2\ x,0,0$; $g_2 = 2\ \frac{1}{2},0,z$; $g_3 = 2\ \frac{1}{2},y,\frac{1}{2}$; $g_4 = 2\ x,x,0$ [the conventions of *International Tables for Crystallography* (1983) are used to designate symmetry operations]. The fifth edge runs along the fourfold rotation axis in $\frac{1}{2},\frac{1}{2},z$ and belongs to type III. No double patches are needed. Generators of U are:

(a) 1 as site symmetry group of the surface patch;

(b) $g_1g_2 = 2\ \frac{1}{2},y,0$
$g_1g_3 = 2(0,0,-1)\ \frac{1}{2},0,z$
$g_1g_4 = 4^-\ 0,0,z$
$g_2g_3 = 2\ x,0,\frac{1}{2}$
$g_2g_4 = 2(\frac{1}{2},-\frac{1}{2},0)\ \frac{1}{2}-x,x,0$
$g_3g_4 = 4^+(0,0,1)\ \frac{1}{2},\frac{1}{2},z$;

(c) –;

(d) $h = 4^-\ \frac{1}{2},\frac{1}{2},z$
$hg_1 = 2(-\frac{1}{2},\frac{1}{2},0)\ \frac{1}{2}-x,x,0$
$hg_2 = 4^+\ 0,0,z$
$hg_3 = 2\ x,x,\frac{1}{2}$
$hg_4 = 2\ x,\frac{1}{2},0$

$\Longrightarrow\ U = I422(\mathbf{a}' = \mathbf{a}-\mathbf{b}, \mathbf{b}' = \mathbf{a}+\mathbf{b}, \mathbf{c}' = 2\mathbf{c})$.

As U is a subgroup of G with index 2, the generated minimal surface subdivides R^3 into two labyrinths with symmetry $I422$.

Example 3 *(cont.)*. Two edges belong to type I; the corresponding twofold rotations are: $g_1 = 2\ x,x,0$; $g_2 = 2\ \frac{1}{2},y,0$. The edges along $\frac{1}{2},y,y$ and $x,\frac{1}{2},x$ belong to type II. Neighbouring surface patches are mapped onto another by $3^-\ x,x,x$ and $3^+\ x,x,x$, respectively. The fifth edge runs along the fourfold axis in $0,y,0$ and belongs to type III. In this case, a double patch is needed: 000, $\frac{1}{2}\frac{1}{2}0$, 100, $1\frac{1}{2}0$, $\frac{1}{2}\frac{1}{2}-\frac{1}{2}$, $\frac{1}{2}00$, $\frac{1}{2}\frac{1}{2}\frac{1}{2}$, $0\frac{1}{2}0$. It has one additional edge for type I ($g_3 = 2\ 1-x,x,0$), two additional edges of type II ($4^+\ \frac{1}{2},\frac{1}{2},z$ and $4^-\ \frac{1}{2},\frac{1}{2},z$) and one additional edge of type III along the fourfold axis in $1,y,0$. Generators of U are:

(a) 1 as site symmetry group of the surface patch;

(b) $g_1g_2 = g_2g_3 = 4^-\ \frac{1}{2},\frac{1}{2},z$
$g_1g_3 = 2\ \frac{1}{2},\frac{1}{2},z$;

(c) $3^+\ x,x,x$, $3^-\ x,x,x$
$4^+\ \frac{1}{2},\frac{1}{2},z$, $4^-\ \frac{1}{2},\frac{1}{2},z$;

(d) $h_1 = 4^+\ 0,y,0$, $h_2 = 4^-\ 1,y,0$
$h_1g_1 = 3^+\ -x,-x,x$
$h_1g_2 = 4^-\ \frac{1}{2},y,-\frac{1}{2}$
$h_1g_3 = 3^+\ \frac{1}{2}+x,\frac{1}{2}-x,-\frac{1}{2}-x$
$h_2g_1 = 3^+\ \frac{1}{2}+x,\frac{1}{2}+x,-\frac{1}{2}+x$
$h_2g_2 = 4^+\ \frac{1}{2},y,-\frac{1}{2}$
$h_2g_3 = 3^+\ \frac{1}{2}-x,\frac{1}{2}+x,-\frac{1}{2}-x$

$\Longrightarrow\ U = 432$ at $\frac{1}{2}\frac{1}{2}-\frac{1}{2}$.

The generated group U is a point group with fixed point at $\frac{1}{2}\frac{1}{2}-\frac{1}{2}$. As a consequence, the minimal surface subdivides R^3 into finite spatial subunits with centres at Wyckoff positions 1(b) of space group $P432$.

3.7. *The Euler characteristic χ of a minimal surface*

Topologically, a non-periodic surface in R^3 may be characterized by three integers (ε, ρ, χ): ε is the orientability character with the value +1 for an orientable and −1 for a non-orientable surface, ρ is the number of boundary curves and χ the 'Euler characteristic'. χ can be derived from any polygon tiling of the surface by

$$\chi = f - e + v, \tag{1}$$

where f is the number of faces (tiles), e the number of edges and v the number of vertices in the tiling. For any periodic surface, the so-defined Euler characteristic becomes necessarily infinite and, therefore, χ is usually referred to one primitive unit cell of such a surface.

In the case of a three-periodic minimal surface generated by a disc-like spanned skew polygon, this polygon yields a tiling on the surface which may be used for the calculation of χ. Regardless of whether or not the surface is self-intersecting, χ is referred to one primitive unit cell of the (eventually oriented) surface. Then f equals the number of skew polygons per unit cell. If the surface intersects itself along a certain edge or in a certain vertex, that edge or that vertex has to be counted more than once in the polygon tiling. Therefore, e and v may be calculated as follows:

$$e = \tfrac{1}{2}mf \tag{2}$$

$$v = f\sum_{i=1}^{m} q_i. \tag{3}$$

Here m is the number of vertices of the generating polygon and q_i depends on the edge angle α_i at the ith vertex, *i.e.* on the number of polygons sharing that vertex (referred to one piece of the surface in the case of self-intersection): $\alpha_i = 30^\circ \Rightarrow q_i = \frac{1}{12}$, $\alpha_i = 45^\circ \Rightarrow q_i = \frac{1}{8}$, $\alpha_i = 60^\circ \Rightarrow q_i = \frac{1}{6}$, $\alpha_i = 90^\circ \Rightarrow q_i = \frac{1}{4}$, *i.e.* $q_i = 360^\circ/\alpha_i$ for these cases. Each 120° vertex, however, corresponds to a branch point and, therefore, six polygons meet at such a vertex: $\alpha_i = 120^\circ \Rightarrow q_i = \frac{1}{6}$.

Equation (4) presents a different possibility for the calculation of χ. It is based on the general Gauss–Bonnet formula (*cf. e.g.* Dierkes *et al.*, 1992, Vol. I):

$$\chi = f - f/(2\pi)\sum_{i=1}^{m}\eta_i - b. \tag{4}$$

Again the summation runs over the m vertices of the generating polygon; $\eta_i = \pi - \alpha_i$ is the exterior edge angle at the ith vertex and b is the number of branch points of the surface per unit cell.

For any three-periodic minimal surface, the Euler characteristic χ is necessarily negative. The larger its absolute value $|\chi|$, the more complicated in the topological sense is that part of the surface that corresponds to one unit cell.

Example 6. $G = P422$; vertices of the generating polygon: 000, $\frac{1}{2}00$, $\frac{1}{2}0\frac{1}{2}$, $10\frac{1}{2}$, $\frac{1}{2}\frac{1}{2}\frac{1}{2}$, $00\frac{1}{2}$; polygon edges: $2(x,0,0)$, $2(\frac{1}{2},0,z)$, $2(x,0,\frac{1}{2})$, $2(1-x,x,\frac{1}{2})$, $2(x,x,\frac{1}{2})$, $2(0,0,z)$. The surface is non-orientable and has no branch points: $f = 8$, $e = \frac{1}{2}\times 6\times 8 = 24$, $v = 8(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{4}+\frac{1}{4}) = 11$. Equation (1) gives $\chi = 8 - 24 + 11 = -5$. Equation (4) gives $\chi = 8 - 8(\pi/2+\pi/2+\pi/2+3\pi/4+\pi/2+\pi/2)/(2\pi) = -5$.

Example 7. $G = P622$; vertices of the generating polygon: 000, $\frac{1}{2}00$, $\frac{1}{2}0\frac{1}{2}$, $10\frac{1}{2}$, $\frac{2}{3}\frac{1}{3}\frac{1}{2}$, $00\frac{1}{2}$; polygon edges: $2(2x,0,0)$, $2(\frac{1}{2},0,z)$, $2(x,0,\frac{1}{2})$, $2(1-x,x,\frac{1}{2})$, $2(2x,x,\frac{1}{2})$, $2(0,0,z)$. The surface is non-orientable and has two branch points per unit cell at $\frac{2}{3}\frac{1}{3}\frac{1}{2}$ and $\frac{1}{3}\frac{2}{3}\frac{1}{2}$: $f = 12$, $e = \frac{1}{2}\times 6\times 12 = 36$, $v = 12(\frac{1}{4}+\frac{1}{4}+\frac{1}{12}+\frac{1}{6}+\frac{1}{4}+\frac{1}{4}) = 15$, $b = 2$. Equation (1) gives $\chi = 12 - 36 + 15 = -9$. Equation (4) gives $\chi = 12 - 12(\pi/2+\pi/2+5\pi/6+\pi/3+\pi/2+\pi/2)/(2\pi) - 2 = -9$.

3.8. *The Euler characteristic* χ_s *of a spatial subunit*

One may also calculate the Euler characteristic χ_s of the surface of a certain labyrinth or of another spatial subunit:

$$\chi_s = f_s - e_s + v_s. \tag{5}$$

In the case of a finite subunit, f_s, e_s and v_s are the numbers of faces, edges and vertices of that polyhedron, respectively. In the case of a labyrinth, a flat labyrinth or a tube, f_s, e_s and v_s are referred to one unit cell of the corresponding space group, layer group or rod group, respectively, *i.e.* f_s is the number of polygons per unit cell which form the boundary of the spatial subunit, and e_s and v_s may be calculated as

$$e_s = \tfrac{1}{2}mf_s \tag{6}$$

and

$$v_s = f_s\sum_{i=1}^{m} 1/p_{si}. \tag{7}$$

Here, p_{si} is the number of polygons sharing the ith vertex of the generating polygon and forming the boundary of the spatial subunit. In contrast to the calculation of χ for the minimal surface, p_{si} cannot be inferred directly from the edge angle α_i at the ith vertex, because here the edge angles of different polygons sharing the same vertex may differ. Therefore, p_{si} must be determined individually for each vertex of the generating polygon by geometrical inspection.

Additional difficulties occur if there exists a polygon edge of type II: Depending on what side of the surface patch is considered, the numbers of polygons sharing the ith vertex may be different, *e.g.* p_{si1} and p_{si2}. If there exist two different kinds of spatial subunits with Euler characteristics χ_{s1} and χ_{s2}, the values p_{si1} and p_{si2} must be used for the calculation of χ_{s1} and χ_{s2}, respectively. If, however, all subunits are congruent, then each of the values p_{si1} and p_{si2} refer to one half of the tiles forming the boundary of a spatial subunit, and equation (7) must be replaced by:

$$v_s = f_s/2\sum_{i=1}^{m}(1/p_{si1} + 1/p_{si2}). \tag{8}$$

Polyhedral spatial subunits without handles have the same Euler characteristic as a sphere, namely $\chi_s = 2$; unbranched tubes have $\chi_s = 0$. The χ_s values for all other spatial subunits are smaller: for branched polyhedra, $\chi_s \le 0$; for branched tubes, $\chi_s \le -2$; for flat labyrinths, $\chi_s \le -2$; for three-periodic labyrinths, $\chi_s \le -4$.

For three-periodic minimal surfaces without self-intersection, $\chi = \chi_s$ holds. In general, this is not true for self-intersecting three-periodic minimal surfaces.

There exists a second possibility for the determination of χ_s, namely with the aid of the 'labyrinth graph' corresponding to each spatial subunit. Such a labyrinth graph has the following properties (*cf.* Fischer & Koch, 1989*c*): (i) each graph is entirely located within its spatial subunit; (ii) each branch of a (flat) labyrinth or of a branched tube or branched polyhedron contains an edge of its labyrinth graph; (iii) each circuit of one labyrinth graph encircles at least one edge of another graph.

As the surface of a spatial subunit is necessarily an orientable surface, its Euler characteristic χ_s and its genus g_s are linked according to the following equation:

$$\chi_s = 2 - 2g_s, \tag{9}$$

where g_s is referred to the same unit cell as χ_s. It may be calculated with the aid of a finite connected subgraph of the regarded labyrinth graph containing no translationally equivalent vertices:

$$g_s = \tfrac{1}{2}s + t. \quad (10)$$

Here s is the number of edges connecting the subgraph to the rest of the infinite labyrinth graph and t is the number of edges that must be omitted to make the subgraph simply connected.

The combination of equations (9) and (10) yields:

$$\chi_s = 2 - s - 2t. \quad (11)$$

Example 6 *(cont.)*. Self-intersection of the surface takes place along edges 4, 5 and 6; edges 4 and 5 belong to type II. The spatial subunits are congruent tubes parallel to **c** with symmetry $U = P4(22)(\mathbf{c}' = \mathbf{c})$ and the axis at $\frac{1}{2}\frac{1}{2}z$. There exists only one tube around each axis. As each surface patch contributes two sides to the surfaces of the spatial subunits, $f_s = 2f = 16$ holds. Equation (6) gives $e_s = \frac{1}{2} \times 6 \times 16 = 48$. Geometrical inspection of the generating polygon shows that, except for vertex 5, each vertex is shared by four surface patches belonging to the same spatial subunit. Vertex 5, however, is common to two or four surface patches depending on the polygon side under consideration. Equation (8) gives $v_s = 16/2(6 \times \frac{1}{4} + 5 \times \frac{1}{4} + \frac{1}{2}) = 26$. Equation (5) gives $\chi_s = 16 - 48 + 26 = -6$. The labyrinth graph has only one vertex per unit cell of $P4(22)$. It is connected *via* four edges to each of its two neighbouring vertices in the unit cells above and below: $s = 8$, $t = 0$. Equation (10) gives $g_s = 8/2 + 0 = 4$. Equation (11) gives $\chi_s = 2 - 8 - 0 = -6$.

Example 7 *(cont.)*. Self-intersection of the surface takes place along edges 4, 5 and 6; edges 4 and 5 belong to type II. The spatial subunits are congruent tubes parallel to **c** with symmetry $U = P3(12)(\mathbf{c}' = \mathbf{c})$ and the axis at $\frac{2}{3}\frac{1}{3}z$. There exists only one tube around each axis, but two tubes per unit cell of G: $f_s = f = 12$. Equation (6) gives $e_s = \frac{1}{2} \times 6 \times 12 = 36$. With the exception of vertex 5, each vertex is shared by four surface patches belonging to the same spatial subunit. Vertex 5, however, is common to two or to three surface patches depending on the polygon side under consideration. Equation (8) gives $v_s = (12/2)(6 \times \frac{1}{4} + 4 \times \frac{1}{4} + \frac{1}{2} + \frac{1}{3}) = 20$. Equation (5) gives $\chi_s = 12 - 36 + 20 = -4$. The labyrinth graph has only one vertex per unit cell of $P3(12)$. It is connected *via* three edges to each of its two neighbouring vertices in the unit cells above and below: $s = 6$, $t = 0$. Equation (10) gives $g_s = 6/2 + 0 = 3$. Equation (11) gives $\chi_s = 2 - 6 - 0 = -4$.

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